Casimir energy of dielectric systems

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A new formula for the Casimir energy of a dispersive dilute dielectric ball is discussed. The formula for the Casimir energy of a polarizable particle situated in a perfectly conducting wedge-shaped cavity is derived by a path-integral coordinate space method in quantum field theory.

1. Introduction

This paper is devoted to study of Casimir effect [1] in dielectrics at zero temperature in the framework of quantum field theory. The presentation of the subject is based on the results and proofs derived by the author.

The paper is organized as follows. In section 2 a new formula for the Casimir energy of a dispersive dilute dielectric ball [2] is discussed. In section 3 a path-integral method is used to justify the formalism that was originally developed by Lifshitz in the framework of statistical physics [3, 4]. This method was applied in Ref. [5] to derive for the first time a formula for the Casimir energy of a polarizable particle situated in a perfectly conducting wedge-shaped cavity. These examples illustrate two different regimes: a dilute connected dielectric with pairwise dipole-dipole interaction between atoms, and the system of two disjoint dielectrics, one of which is not a dilute one, where many-body effects of non-pairwise interaction are important.

We put $\hbar = c = 1$ and use rationalized Gaussian units where the polarizability of atoms $\alpha(i\omega)$ is defined via $\varepsilon(i\omega) - 1 = 4\pi\rho\,\alpha(i\omega)$, ρ is a number density of atoms, ω_0 is a characteristic absorption frequency of materials, λ is an average minimum distance between atoms of a dielectric, Casimir–Polder potential is defined by $-23\,\alpha_1(0)\alpha_2(0)/4\pi r^7$.

2. Casimir energy of a dispersive dilute dielectric ball

We study a dielectric nonmagnetic ball of the radius a and permittivity ε, surrounded by a vacuum. The ball is dilute, i.e. all final expressions are obtained under the assumption $\varepsilon - 1 \ll 1$ in the order $(\varepsilon(i\omega) - 1)^2$, the lowest order that yields the energy of interaction between atoms of the ball.

The study of the Casimir energy of a nonmagnetic dielectric ball remains one of the main problems in the theory of Casimir effect. Cut-off dependent terms arised in every macroscopic approach to the problem, so it was possible to extract correctly only the large distance contribution to the energy of a dilute dielectric ball by use of macroscopic methods (see Refs. [6], [7] and Appendix in Ref.[8]). The regularization of ill-defined expressions remains the main problem of various macroscopic approaches to connected dielectrics. Usually the Casimir energy of a disjoint macroscopic system (two dispersive dielectric parallel plates is a classic example by Lifshitz) depends only on the distance between macroscopic bodies and dispersion of dielectrics [4]. On the other hand, it was argued in Refs. [8], [9] and [2] that for a dilute connected dielectric the Casimir energy is equal to the energy of dipole-dipole pairwise interactions of all atoms constituting the dielectric and thus should also depend on an average minimum distance between atoms of a dielectric λ. For a dilute dispersive dielectric ball with an arbitrary frequency dependent dielectric permittivity the Casimir energy was first derived

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in Ref. [2]:

$$E = -\rho^2 \frac{\pi}{48} \int_0^{+\infty} d\omega \alpha^2(i\omega)$$

$$\left(\frac{a^3}{\lambda^3} e^{-2\omega\lambda}(128 + 256\omega\lambda + 128\omega^2\lambda^2 + 64\omega^3\lambda^3)\right.$$

$$-\frac{a^2}{\lambda^2}\left(e^{-2\omega\lambda}(144 + 288\omega\lambda + 120\omega^2\lambda^2 + 48\omega^3\lambda^3)\right.$$

$$\left.-96\omega^2\lambda^2 E_1(2\omega\lambda)\right)$$

$$+\left(e^{-2\omega\lambda}(41 + 34\omega\lambda + 14\omega^2\lambda^2 + 4\omega^3\lambda^3)\right. \qquad (1)$$

$$+24E_1(2\omega\lambda)\big)+$$

$$\left. \left(e^{-4\omega a}(-21 + 12\omega a) - E_1(4\omega a)(24 + 96\omega^2 a^2)\right)\right).$$

Here $E_1(x) = \int_1^{+\infty} e^{-tx}/t\, dt$.

This energy is finite and physical only when a finite separation between atoms λ is taken into account. In the hypothetical non-physical limit $\lambda \to 0$ the leading term in (1) (V is a ball volume) is

$$-\rho^2 \frac{2V}{\lambda^3} \int_0^{+\infty} d\omega\, \alpha^2(i\omega), \qquad (2)$$

so in the limit $\lambda \to 0$ the Casimir energy is divergent for every model of atomic polarizability $\alpha(i\omega)$.

The formula (1) finally solves the problem of the Casimir energy for a dilute dielectric ball. The interest to this topic strongly arised after the series of articles by Schwinger [10] where he tried to treat the sonoluminescence of bubbles in water [11] as a dynamical Casimir effect. Schwinger suggested that the first order terms $\sim (\varepsilon - 1)V$ should yield the main contribution to the energy of the ball. One possible argument why the first order terms have no influence on physics is the condition of the conservation of atoms constituting the ball:

$$(\varepsilon - 1)V = \text{const.} \qquad (3)$$

It follows from this condition that the first order terms do not change their value during the ball collapse or expansion, so they can be subtracted from the energy.

After publication of the articles [6] it was generally believed that the Casimir energy of a dilute dielectric ball is equal to (this expression was first derived in [12])

$$E_{ld} = \frac{23}{1536\pi a}(\varepsilon - 1)^2, \qquad (4)$$

which is only a non-dispersive limit of the last line of our new full expression for the energy (1). To check this, we write the leading contribution from the last line of (1) as (it comes from frequencies $\omega \ll \omega_0$, ω_0 is a characteristic absorption frequency of materials, $\omega_0 a \gg 1$, so it is possible to use the static polarizability $\alpha(0)$ in the leading approximation to the last line of (1))

$$-\rho^2\alpha^2(0)\frac{\pi}{48} \int_0^{+\infty} d\omega \left(e^{-4\omega a}(-21 + 12\omega a)\right.$$

$$-E_1(4\omega a)(24 + 96\omega^2 a^2))$$

$$= \rho^2\alpha^2(0)\frac{23}{96}\frac{\pi}{a} = \frac{23}{1536\pi a}(\varepsilon - 1)^2 = E_{ld}. \qquad (5)$$

The new full energy expression contains additional terms: volume and surface contributions to the energy, as well as the terms which do not depend on the ball radius a.

The term (4) can be called a large distance contribution to the Casimir energy of the ball. However, it is impossible to separate large distances between atoms of a dielectric ball from short distances between atoms of the ball in any possible experiment. This is why only the use of a dipole-dipole potential valid for all existing distances between atoms of the ball (which are all greater than λ) is physically reasonable for the calculation of Casimir energies of dilute bodies. To understand this, consider the line of the following examples. Casimir energy of two neutral atoms coincides with the energy of a dipole-dipole interaction of these atoms. When an atom is located outside a dielectric of an arbitrary form, then in a dilute approximation the Casimir energy is equal to the sum of dipole-dipole interactions between this atom and atoms of the dielectric [8]. For two parallel dielectric slabs it is known that the Casimir energy in a dilute approximation is equal to the sum of pairwise dipole-dipole interactions of atoms constituting the slabs. For a dilute dielectric ball the sum of pairwise dipole-dipole interactions of atoms constituting the ball is given by Eq.(1), not Eq.(4).

There are several important differences between our microscopic calculation [2] and microscopic calculation in [12] where the term (4) was first derived. In the article [12] the Casimir-Polder potential, which is a large distance limit of a dipole-dipole potential, valid only for distances $r \gg 1/\omega_0$, was used in the calculation of the ball Casimir energy. The calculation in Ref. [12] was non-dispersive from the outset and thus could not yield the contribution to the energy from short distances ($r < 1/\omega_0$) between atoms of the ball. Moreover, the use of a dimensional regularization in Ref. [12] concealed the divergences which would appear in the energy expression from the integration over short distances between atoms in the 3-dimensional ball since the minimum distance between atoms of the ball was not introduced in Ref. [12].

Needless to say that the term (4) itself was really important for development of the theory of Casimir effect in connected dielectrics since this term has been derived via different techniques [12, 6]. However, using these approaches one can extract correctly only the large distance contribution to the Casimir energy of the ball, e.g. E_{ld}. This large distance contribution E_{ld} was found to be the same when summing up the Casimir–Polder potential between atoms of the ball [12] and when the Casimir energy was derived by field-theoretic calculations [6] - so the equivalence of large distance parts of the Casimir energy for a dilute dielectric ball derived by microscopic and macroscopic approaches was proved.

It is important to stress that so far macroscopic methods did not yield satisfactorily short distance contributions to the Casimir energy of connected dielectrics. The reason is simple: these methods were developed for *disjoint*, not connected dielectrics, and application of these methods to connected dielectrics without any changes inevitably leads to different types of divergences in *every* field-theoretic calculation of the Casimir energy. These divergences are reminiscents of the ill-defined short distance structure of the theory.

It was generally believed that Casimir surface force is repulsive before the appearance of our papers [8, 2]. So it is natural to give here a proof that Casimir surface force on a dilute dielectric ball is attractive.

It is convenient to define $N \equiv a/\lambda, p \equiv \omega\lambda$. Then Eq.(1) can be rewritten in a general form

$$E = -\frac{\rho^2}{\lambda} \int_0^{+\infty} dp\, \alpha^2 \left(i\frac{p}{\lambda}\right) f(N, p). \qquad (6)$$

The function $f(N, p) > 0$ for $N > 1/2, p > 0$. The ball expands or collapses homogeneously, so

$$N = \text{const}. \qquad (7)$$

Conservation of atoms inside the ball imposes the condition

$$\rho \frac{4\pi a^3}{3} = const. \qquad (8)$$

It is convenient to use Kramers–Kronig relations in the form

$$\alpha(i\omega) = \int_0^{+\infty} dx\, \frac{xg(x)}{x^2 + \omega^2}, \qquad (9)$$

where the condition $g(x) > 0$ always holds. Using (6), (7), (8), (9), Casimir force on a unit surface is equal to

$$F = -\frac{1}{4\pi a^2}\frac{\partial E}{\partial a} = -\frac{\rho^2}{4\pi a^3} \int_0^{+\infty} d\omega \int_0^{+\infty} dx$$
$$\frac{x(7x^2 + 3\omega^2)g(x)}{(x^2 + \omega^2)^2}\, \alpha(i\omega)f(N, \omega\lambda) < 0. \qquad (10)$$

$F < 0$ because all functions inside integrals are positive. Casimir surface force is attractive for every model of atomic polarizability consistent with general causal requirements.

3. Casimir energy of a polarizable particle in a perfectly conducting wedge

The one-loop effective action, which is the result of the integration over quantum fluctuations of the electromagnetic field, has the form

$$W = \frac{1}{2}\text{Tr}\ln L_{pp}(\varepsilon, t, \mathbf{x}), \qquad (11)$$

where

$$L_{jm}(\varepsilon, \omega, \mathbf{r}) = \left[\varepsilon(i|\omega|, \mathbf{r})\,\omega^2\delta_{jm} + \text{rot}_{jl}\text{rot}_{lm}\right]. (12)$$

Electromagnetic field propagator $D_{mk}(\varepsilon, \omega, \mathbf{r}, \mathbf{r}')$ in a gauge $A_0 = 0$ satisfies the equations

$$L_{jm}(\varepsilon, \omega, \mathbf{r})D_{mk}(\varepsilon, \omega, \mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')\delta_{jk}. \qquad (13)$$

4

Imagine that the space was empty at first, with no dielectric in it. Imagine then that the particle with an atomic polarizability $\alpha(i|\omega|)$ is inserted at the point $\mathbf{x} = \mathbf{r}'$ so that $\delta\varepsilon(i|\omega|, \mathbf{x}) = 4\pi\alpha(i|\omega|)\,\delta^3(\mathbf{x} - \mathbf{r}')$. The change in the ground state energy for a small $\alpha(i|\omega|)$ is equal to

$$\delta E_1 = \int_{-\infty}^{+\infty} d\omega\, \alpha(i|\omega|)\,\omega^2 D_{pp}(\varepsilon = 1, \omega, \mathbf{r}', \mathbf{r}').\ (14)$$

This change is divergent. However, there are no other particles around and thus this is not the energy of interaction that can be measured.

Imagine now that the same particle is inserted in the neighbourhood of a dielectric body with an arbitrary permittivity $\varepsilon(i|\omega|, \mathbf{r_2})$. The change in the ground state energy in this case is given by

$$\delta E_2 = \int_{-\infty}^{+\infty} d\omega\, \alpha(i|\omega|)\,\omega^2 D_{pp}(\varepsilon, \omega, \mathbf{r}', \mathbf{r}'),\quad (15)$$

but this is not the answer for the energy of interaction. The energy responsible for interaction of a particle and a dielectric is finite and equal to $\delta E = \delta E_2 - \delta E_1$:

$$\delta E = \int_{-\infty}^{+\infty} d\omega\, \alpha(i|\omega|)\,\omega^2 (D_{pp}(\varepsilon, \omega, \mathbf{r}'', \mathbf{r}')$$
$$- D_{pp}(\varepsilon = 1, \omega, \mathbf{r}'', \mathbf{r}'))\Big|_{\mathbf{r}'' \to \mathbf{r}'}. \quad (16)$$

This formula can be used to calculate the Casimir energy of a polarizable particle with the polarizability $\alpha(i\omega)$ located at the point with cylindrical coordinates (r, θ, z) in a perfectly conducting wedge-shaped cavity [5] (the walls of the wedge have coordinates $(r, 0, z)$ and (r, α, z), $0 < \theta < \alpha$).

This system may be described by the set of equations (13) with $\varepsilon = 1$ outside the wedge walls and perfect boundary conditions imposed on each spatial argument of $D_{ij}(\varepsilon = 1, \omega, \mathbf{r}, \mathbf{r}')$ at the wedge walls.

For distances $r\theta, r(\alpha - \theta) \gg \lambda_0 \sim 50$nm one can neglect dispersion in an atomic polarizability of the particle and walls of the wedge and take the limit $\alpha(0)$ from the beginning. Perfectly conducting walls of the wedge can be considered as the limiting case of the walls with a constant permittivity $\varepsilon(0)$ when $\varepsilon(0) \to \infty$.

Casimir energy of a polarizable particle situated in a perfectly conducting wedge-shaped cavity was first calculated in Ref. [5]. It can be derived from (16) in the form:

$$\varepsilon(\mathbf{r}) = -\frac{\alpha(0)}{4\pi r^4}\left[\frac{3}{2}\frac{p^4}{\sin^4 p\theta} - \frac{p^2(p-1)(p+1)}{\sin^2 p\theta}\right.$$
$$\left. -\frac{1}{90}(p-1)(p+1)(p^2+11)\right], \quad (17)$$

where $p = \frac{\pi}{\alpha}$.

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